

Existence and multiplicity results for boundary value problems connected with the discrete $p(\cdot)$ –Laplacian on weighted finite graphs

Marek Galewski and Renata Wieteska

Abstract

We use the direct variational method, the Ekeland variational principle, the mountain pass geometry and Karush-Kuhn-Tucker theorem in order to investigate existence and multiplicity results for boundary value problems connected with the discrete $p(\cdot)$ –Laplacian on weighted finite graphs. Several auxiliary inequalities for the discrete $p(\cdot)$ –Laplacian on finite graphs are also derived. Positive solutions are considered.

Keywords: weighted graph; $p(\cdot)$ –Laplacian on a graph; critical point theory; existence and multiplicity

1. Introduction

In this note we will consider the following boundary value problem, namely

$$\begin{cases} -\Delta_{p(x),\omega}u(x) + q(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x, u(x)), & x \in S, \\ u(x) = 0, & x \in \partial S. \end{cases} \quad (1.1)$$

where $\overline{S} = (S \cup \partial S, V)$ is a simple, connected, undirected and weighted graph with two finite, disjoint and nonempty sets S and ∂S of vertices, called interior and boundary, respectively, and with a set V of unordered pairs of distinct elements of $S \cup \partial S$ whose elements are called edges, $\omega : \overline{S} \times \overline{S} \rightarrow [0, +\infty)$ is a weight on a graph \overline{S} , $u : \overline{S} \rightarrow \mathbb{R}$, $q : S \rightarrow (0, +\infty)$, $p : \overline{S} \rightarrow [2, +\infty)$ are discrete functions, $f : S \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\Delta_{p(\cdot),\omega}$ is the discrete $p(\cdot)$ –Laplacian defined on a graph and λ is a real positive parameter. The continuity of a function f means that for any fixed $x \in S$ the function $f(x, \cdot)$ is continuous.

The aim of this paper is to investigate the existence and multiplicity of positive solutions for problem (1.1) applying mainly variational methods, such as the direct variational method, the Ekeland variational principle, the mountain pass geometry and Karush-Kuhn-Tucker theorem. Several auxiliary inequalities for the discrete $p(\cdot)$ –Laplacian on finite graphs are also derived. Positive

solutions are the only one that have physical meaning. That is why only these are considered.

We note that there is a big difference between discrete problems with the p -Laplacian and their graph counterparts especially in the case when the graph is weighted which is the case of this paper. For the discrete problem, the potential of the p -Laplacian, the so called isotropic case, is coercive and has well recognized relations with any norm we can choose on the underlying space. For the p -Laplacian on the weighted graph there are no such simple relations contrary to the case of problems on non-weighted graphs which behave almost like their discrete counterparts. Direct calculations can be performed in order to ascertain about that. The above mentioned reasons require considering the term $q(x)|u(x)|^{p-2}u(x)$ which is crucial if one wishes to apply critical point theory and variational methods. The relations between this term and the nonlinearity will allow us to apply classical variational tools mentioned further in Section 2.

We would like to underline that to the best of our knowledge, the discrete $p(\cdot)$ -Laplacian on finite graphs have not been considered yet. This means that we had to investigate the problem in a detailed manner which involves derivation of many auxiliary inequalities that are necessary for having the relation between the norm and potential of the graph $p(\cdot)$ -Laplacian, since now we work in the anisotropic case. The results in the literature cover only the case of the p -Laplacian on graphs, see [20, 21], where however other methods are applied. Thus our results are new also in the context of constant p . Problems with the graph $p(\cdot)$ -Laplacian called anisotropic boundary value problems are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [26]), electrorheological fluids (see [25]) or image restoration (see [6]). Variational continuous anisotropic problems have been started by Fan and Zhang in [9] and later considered by many methods and authors - see [12] for an extensive survey of such boundary value problems. In the discrete setting see for example [2], [14], [17], [22] for the most recent results. For a background on variational methods we refer to [16, 24] while for a background on difference equations to [1].

We would like to note that we improve here some the existence and multiplicity results obtained in [10] for a discrete boundary value problem, see for example Theorem 5.11 which in [10] requires some assumption on the growth of nonlinearity which is not assumed here. As concerns Theorem 6.13 one can derive its counterpart for the problem considered in [10].

The paper is organized as follows. Firstly, after providing basic information about the graph theoretic notions, we recall the fundamental tools from critical point theory, which cover the Weierstrass Theorem, the Mountain Pass Lemma, the Ekeland Variational Principle and also Karush-Kuhn-Tucker Theorem. Next we provide several inequalities useful in variational investigations of our problem. Then we give a variational formulation of the considered problem. The existence of positive solutions is investigated in the last section. In the first step we establish conditions under which we can obtain the existence of at least one positive solution. We apply the direct variational method and Ekeland variational principle. In the second step we are interested in the mul-

tiplicity of positive solutions. Using the mountain pass technique both with the Ekeland variational principle and Karush-Kuhn-Tucker conditions we obtain the existence of at least two distinct positive solutions. The examples are also provided.

2. Preliminary results

In this section we provide some tools which are used throughout the paper. We start with providing basic information about the graph theoretic notions.

Let $\overline{S} = (S \cup \partial S, V)$ be a simple, connected, undirected and weighted graph. A weight on a graph \overline{S} is a function $\omega : \overline{S} \times \overline{S} \rightarrow [0, +\infty)$ satisfying

- (i) $\omega(x, y) = \omega(y, x)$ if $\{x, y\} \in V$;
- (ii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin V$.

Let $u : \overline{S} \rightarrow \mathbb{R}$ and $p : \overline{S} \rightarrow (1, +\infty)$. The $p(\cdot)$ -gradient $\nabla_{p(\cdot), \omega}$ of the function u is defined by

$$\nabla_{p(x), \omega} u(x) := (D_{p(x), \omega, y} u(x))_{y \in \overline{S}} \quad \text{for all } x \in \overline{S},$$

where

$$D_{p(x), \omega, y} u(x) := |u(y) - u(x)|^{p(x)-2} (u(y) - u(x)) \sqrt{\omega(x, y)} \quad \text{for all } x \in \overline{S}$$

there is the $p(\cdot)$ -directional derivative of the function u in the direction y . In case of $p(\cdot) \equiv 2$ we write ∇_ω .

The discrete $p(\cdot)$ -Laplacian $\Delta_{p(\cdot), \omega}$ of the function u is defined by

$$\Delta_{p(x), \omega} u(x) := \sum_{y \in \overline{S}} |u(y) - u(x)|^{p(x)-2} (u(y) - u(x)) \omega(x, y) \quad \text{for all } x \in \overline{S}.$$

The integration of the function u on a graph \overline{S} is defined by

$$\int_{\overline{S}} u := \sum_{x \in \overline{S}} u(x).$$

For any pair of functions $u, v : \overline{S} \rightarrow \mathbb{R}$ we have by direct calculation (see [7, 15])

$$2 \int_{\overline{S}} (-\Delta_{p, \omega} u) v = \int_{\overline{S}} \nabla_{p, \omega} u \circ \nabla_\omega v. \quad (2.2)$$

Now we recall the fundamental tools from critical point theory. Let $(E, \|\cdot\|)$ be a real Banach space and let $J : E \rightarrow \mathbb{R}$. We say that the functional J is coercive if $\lim_{\|u\| \rightarrow \infty} J(u) = +\infty$ and anticoercive if $\lim_{\|u\| \rightarrow \infty} J(u) = -\infty$.

Theorem 2.1 [18] (*Weierstrass Theorem*) *Let E be a reflexive Banach space. If a functional $J \in C^1(E, \mathbb{R})$ is weakly lower semi-continuous and coercive then there exists $\bar{x} \in E$ such that $\inf_{x \in E} J(x) = J(\bar{x})$ and \bar{x} is also a critical point of J , i.e. $J'(\bar{x}) = 0$. Moreover, if J is strictly convex, then a critical point is unique.*

We say ([13]) that a continuously differentiable functional J defined on E satisfies the Palais-Smale condition if every sequence $\{u_n\}$ in E such that

$$\{J(u_n)\} \text{ is bounded and } J'(u_n) \longrightarrow 0 \text{ in } E^* \text{ as } n \longrightarrow \infty$$

has a convergent subsequence.

In this paper we apply the following version of the mountain pass lemma.

Lemma 2.2 [13] (*Mountain Pass Lemma*) *Let E be a real Banach space and let $J \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that there exist $u_0, u_1 \in E$ and a bounded open neighborhood Ω of u_0 such that $u_1 \notin \overline{\Omega}$ and*

$$\max\{J(u_0), J(u_1)\} < \inf_{u \in \partial\Omega} J(u).$$

Let

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)).$$

Then c is a critical value of J ; that is, there exists $u^* \in E$ such that $J'(u^*) = 0$ and $J(u^*) = c$, where $c > \max\{J(u_0), J(u_1)\}$.

We also apply the weak form of Ekeland's variational principle, namely

Theorem 2.3 [8] (*Ekeland's Variational Principle - weak form*) *Let (X, d) be a complete metric space. Let $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below. Then given $\varepsilon > 0$ there exists $u_\varepsilon \in X$ such that:*

$$(1) \Phi(u_\varepsilon) \leq \inf_{u \in X} \Phi(u) + \varepsilon,$$

$$(2) \Phi(u_\varepsilon) < \Phi(u) + \varepsilon d(u, u_\varepsilon) \text{ for all } u \in X \text{ with } u \neq u_\varepsilon.$$

Finally, let us recall Karush-Kuhn-Tucker conditions

Theorem 2.4 [11] (*Karush-Kuhn-Tucker Theorem*) *Let E be a Banach space and J_0, \dots, J_n be functionals on E . Let x_0 be a solution of the problem*

$$\begin{cases} \min_{x \in E} J_0(x); \\ J_i(x) \leq 0, \quad i = 1, \dots, n, \end{cases}$$

and assume that the functionals J_0, \dots, J_n are Fréchet differentiable at x_0 . Then there exist nonnegative real numbers $\lambda_0, \dots, \lambda_n$, not all zero, such that

$$\lambda_i J_i'(x_0) = 0, \quad i = 1, \dots, n,$$

and

$$\lambda_0 J_0'(x_0) + \lambda_1 J_1'(x_0) + \dots + \lambda_n J_n'(x_0) = 0.$$

If the set $\{x : J_i(x) \leq 0, \quad i = 1, \dots, n\}$ has a non-empty interior (Slater's condition) then we can fix $\lambda_0 = 1$.

In this paper we also use ideas in [3], where the Authors proved that if $X \subset E$ is an open set, the functional J satisfies the Palais-Smale condition and

$$\inf_{x \in \overline{X}} J(x) < \inf_{x \in \partial X} J(x) \quad (2.3)$$

then there exists some $x_0 \in E$ which is a critical point to J such that

$$\inf_{x \in \overline{X}} J(x) = J(x_0).$$

The multiplicity result in [3] is also obtained with the aid of the mountain pass lemma in the following context. If X is an open ball centered at 0 with radius $r > 0$, the functional J satisfies the Palais-Smale condition and $J(0) = 0$ then there exists an element $e \in E \setminus X$ such that $J(e) \leq 0$. If additionally

$$-\infty < \inf_{x \in \overline{X}} J(x) < 0 < \inf_{x \in \partial X} J(x),$$

then J has two critical points.

By using the above mentioned methods the Author in [23] obtains the existence of at least two non-zero solutions for some periodic and Neumann problems with the discrete $p(\cdot)$ -Laplacian.

3. Auxiliary inequalities

Recall that in this paper we examine the existence of solutions for problem with Dirichlet boundary conditions. Therefore let us define the space A in which the problem will be considered as follows

$$A := \{u : \overline{S} \rightarrow \mathbb{R} : u(x) = 0 \text{ for all } x \in \partial S\}.$$

Then A is a finite dimensional Euclidean space provided with the norm given by

$$\|u\| := \left(\int_{\overline{S}} |u|^2 \right)^{\frac{1}{2}} \quad (3.4)$$

and with naturally associated scalar product. The dimension of A is $|S|$.

Let us also introduce a notation used throughout the paper, namely

$$p^- := \min_{x \in S} p(x); \quad p^+ := \max_{x \in S} p(x);$$

$$\overline{p}^- := \min_{x \in \overline{S}} p(x); \quad \overline{p}^+ := \max_{x \in \overline{S}} p(x);$$

$$\overline{\omega}^+ := \max_{(x,y) \in \overline{S} \times \overline{S}} \omega(x,y).$$

On the space A the following inequalities hold.

Lemma 3.5

(a.1) For every $u \in A$ and for every $m \geq 1$ we have

$$\sum_{x \in S} |u(x)|^m \leq |S| \|u\|^m.$$

(a.2) For every $u \in A$ and for every $m \geq 2$ we have

$$\sum_{x, y \in \overline{S}} |u(y) - u(x)|^m \leq 2^m |\overline{S}| |S| \|u\|^m.$$

(a.3) For every $u \in A$ and for every $m \geq 2$ we have

$$\sum_{x \in S} |u(x)|^m \geq 2^{-\frac{m}{2}} |\partial S|^{\frac{m}{2}} |\overline{S}|^{1-m} \|u\|^m.$$

(a.4) For every $u \in A$ and for every $p^- \geq 2$ we have

$$\sum_{x \in S} |u(x)|^{p(x)} \geq 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-} \|u\|^{p^-} - |S|.$$

(a.5) For every $u \in A$ and for every $\overline{p}^+ \geq 2$ we have

$$\sum_{x, y \in \overline{S}} |u(y) - u(x)|^{p(x)} \omega(x, y) \leq \overline{\omega}^+ 2^{\overline{p}^+} |\overline{S}| |S| \|u\|^{\overline{p}^+} + \overline{\omega}^+ |\overline{S}|^2.$$

(a.6) For every $u \in A$ and for every $p^+ \geq 2$ we have

$$\sum_{x \in S} |u(x)|^{p(x)} \leq |S| \|u\|^{p^+} + |S|.$$

(a.7) For every $u \in A$ we have

$$\max_{x \in S} |u(x)| \leq |\overline{S}|^{\frac{1}{2}} \|u\|.$$

Proof. We will show that (a.1) holds. For all $x \in S$ we have

$$|u(x)|^2 \leq \sum_{s \in S} |u(s)|^2 = \sum_{s \in \overline{S}} |u(s)|^2.$$

Thus for every $m \geq 1$ we get

$$|u(x)|^m \leq \left(\sum_{s \in \overline{S}} |u(s)|^2 \right)^{\frac{m}{2}},$$

which leads to

$$\sum_{x \in S} |u(x)|^m \leq |S| \|u\|^m.$$

To see **(a.2)** first note that for every $u \in A$ and for every $m > 0$ we have

$$\sum_{x,y \in \bar{S}} |u(x)|^m = \sum_{y \in \bar{S}} \left(\sum_{x \in \bar{S}} |u(x)|^m \right) = |\bar{S}| \sum_{x \in S} |u(x)|^m.$$

Recall also that for every $m \geq 2$ the following inequality hold (see [5])

$$|a+b|^m + |a-b|^m \leq 2^{m-1} (|a|^m + |b|^m) \text{ for all } a, b \in \mathbb{R}.$$

Thus for every $u \in A$ and for every $m \geq 2$ we have

$$\sum_{x,y \in \bar{S}} |u(y) - u(x)|^m \leq 2^m |\bar{S}| \sum_{x \in S} |u(x)|^m. \quad (3.5)$$

In a consequence, by **(a.1)** we get **(a.2)**.

We will show that **(a.3)** holds. Using twice the discrete Hölder inequality we have

$$\begin{aligned} \sum_{x,y \in \bar{S}} |u(y) - u(x)|^2 &= \sum_{x,y \in \bar{S}} |u(y)|^2 + \sum_{x,y \in \bar{S}} |u(x)|^2 - \\ &2 \sum_{x,y \in \bar{S}} (u(y)u(x)) = 2|\bar{S}| \|u\|^2 - 2 \sum_{x \in S} \left(\left(\sum_{y \in S} u(y) \right) u(x) \right) \geq \\ &2|\bar{S}| \|u\|^2 - 2 \left(\sum_{x \in S} \left(\sum_{y \in S} u(y) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{x \in S} |u(x)|^2 \right)^{\frac{1}{2}} = \\ &2|\bar{S}| \|u\|^2 - 2|S|^{\frac{1}{2}} \sum_{y \in S} u(y) \|u\| \geq 2\|u\|^2 |\partial S|. \end{aligned}$$

On the other hand for every $m \geq 2$ the discrete Hölder inequality implies

$$\begin{aligned} \sum_{x,y \in \bar{S}} |u(y) - u(x)|^2 &\leq \left(\sum_{x,y \in \bar{S}} 1^{\frac{m}{m-2}} \right)^{\frac{m-2}{m}} \left(\sum_{x,y \in \bar{S}} \left(|u(y) - u(x)|^2 \right)^{\frac{m}{2}} \right)^{\frac{2}{m}} = \\ &|\bar{S}|^{\frac{2(m-2)}{m}} \left(\sum_{x,y \in \bar{S}} |u(y) - u(x)|^m \right)^{\frac{2}{m}}. \end{aligned}$$

The above inequalities lead to

$$2 \|u\|^2 |\partial S| \leq |\bar{S}|^{\frac{2(m-2)}{m}} \left(\sum_{x,y \in \bar{S}} |u(y) - u(x)|^m \right)^{\frac{2}{m}}.$$

Thus for every $u \in A$ and for every $m \geq 2$ we have

$$\sum_{x,y \in \bar{S}} |u(y) - u(x)|^m \geq 2^{\frac{m}{2}} |\partial S|^{\frac{m}{2}} |\bar{S}|^{2-m} \|u\|^m. \quad (3.6)$$

Combining (3.5) and (3.6) we get **(a.3)**.

Relation **(a.4)** is obtained by **(a.3)** as follows

$$\begin{aligned} \sum_{x \in S} |u(x)|^{p(x)} &\geq \sum_{\{x \in S: |u(x)| > 1\}} |u(x)|^{p^-} + \sum_{\{x \in S: |u(x)| \leq 1\}} |u(x)|^{p^+} = \\ &\sum_{x \in S} |u(x)|^{p^-} - \sum_{\{x \in S: |u(x)| \leq 1\}} \left(|u(x)|^{p^-} - |u(x)|^{p^+} \right) \geq \end{aligned}$$

$$2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\bar{S}|^{1-p^-} \|u\|^{p^-} - |S|.$$

Relation **(a.5)** is obtained by **(a.2)** as follows

$$\begin{aligned} \sum_{x,y \in \bar{S}} |u(y) - u(x)|^{p(x)} \omega(x,y) &\leq \bar{\omega}^+ \sum_{x,y \in \bar{S}} |u(y) - u(x)|^{p(x)} \leq \\ &\bar{\omega}^+ \sum_{\{x,y \in \bar{S}: |u(y) - u(x)| > 1\}} |u(y) - u(x)|^{\bar{p}^+} + \\ &\bar{\omega}^+ \sum_{\{x,y \in \bar{S}: |u(y) - u(x)| \leq 1\}} |u(y) - u(x)|^{\bar{p}^-} \leq \\ &\bar{\omega}^+ \sum_{x,y \in \bar{S}} |u(y) - u(x)|^{\bar{p}^+} + \\ &\bar{\omega}^+ \sum_{\{x,y \in \bar{S}: |u(y) - u(x)| \leq 1\}} \left(|u(y) - u(x)|^{\bar{p}^-} - |u(y) - u(x)|^{\bar{p}^+} \right) \leq \\ &\bar{\omega}^+ 2^{\bar{p}^+} |\bar{S}| |S| \|u\|^{\bar{p}^+} + \bar{\omega}^+ |\bar{S}|^2. \end{aligned}$$

The inequality **(a.6)** we obtain in the same manner as **(a.5)**, using **(a.1)** instead of **(a.2)**.

And finally, the discrete Hölder inequality implies **(a.7)**. Indeed, for every $x \in S$ we have

$$|u(x)| \leq \sum_{s \in S} |u(s)| = \sum_{s \in \overline{S}} |u(s)| \leq |\overline{S}|^{\frac{1}{2}} \|u\|.$$

Therefore

$$\max_{x \in S} |u(x)| \leq |\overline{S}|^{\frac{1}{2}} \|u\|.$$

The proof of Lemma 3.5 is complete. ■

4. Variational framework

In order to study the problem considered we will start with putting in the nonlinear term f the non-negative part of u instead of u . Then we obtain the following boundary value problem

$$\begin{cases} -\Delta_{p(x), \omega} u(x) + q(x) |u(x)|^{p(x)-2} u(x) = \lambda f(x, u_+(x)), & x \in S, \\ u(x) = 0, & x \in \partial S. \end{cases} \quad (4.7)$$

Let us define the functional $J : A \rightarrow \mathbb{R}$ by the following formula

$$J(u) = \frac{1}{2} \int_{\overline{S}} \frac{1}{p} \nabla_{p, \omega} u \circ \nabla_{\omega} u + \int_S \frac{1}{p} q |u|^p - \lambda \int_S F_{u_+}, \quad (4.8)$$

where $F_{u_+} : S \rightarrow \mathbb{R}$ is defined by

$$F_{u_+}(x) = F(x, u_+(x)) := \int_0^{u_+(x)} f(x, s) ds.$$

The functional J can be rewritten as follows

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{x \in \overline{S}} \left(\frac{1}{p(x)} \sum_{y \in \overline{S}} |u(y) - u(x)|^{p(x)} \omega(x, y) \right) \\ &\quad + \sum_{x \in S} \frac{1}{p(x)} q(x) |u(x)|^{p(x)} - \lambda \sum_{x \in S} F(x, u_+(x)) \end{aligned}$$

and we will use both notations when necessary.

We will show that critical points of the functional J correspond to the solutions of problem (4.7).

Theorem 4.6 *The point $u \in A$ is a critical point to J if and only if it satisfies (4.7).*

Proof. Take an arbitrary $u \in A$. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\varphi(\varepsilon) = J(u + \varepsilon v)$, where $v \in A$ is a fixed nonzero direction. Then

$$\begin{aligned} \varphi'(\varepsilon) &= \frac{1}{2} \sum_{x,y \in \overline{S}} |(u + \varepsilon v)(y) - (u + \varepsilon v)(x)|^{p(x)-2} \\ &\quad ((u + \varepsilon v)(y) - (u + \varepsilon v)(x))(v(y) - v(x))\omega(x, y) + \\ &\quad \sum_{x \in S} q(x) |(u + \varepsilon v)(x)|^{p(x)-2} (u + \varepsilon v)(x)v(x) - \lambda \sum_{x \in S} f(x, (u + \varepsilon v)(x))v(x). \end{aligned}$$

Letting $\varepsilon = 0$ we have

$$\begin{aligned} \varphi'(0) &= \frac{1}{2} \sum_{x,y \in \overline{S}} |u(y) - u(x)|^{p(x)-2} (u(y) - u(x))(v(y) - v(x))\omega(x, y) + \\ &\quad \sum_{x \in S} q(x) |u(x)|^{p(x)-2} u(x)v(x) - \lambda \sum_{x \in S} f(x, u_+(x))v(x). \end{aligned} \tag{4.9}$$

Thus by (2.2)

$$\varphi'(0) = \int_{\overline{S}} (-\Delta_{p,\omega} u)v + \int_S q |u|^{p-2} uv - \lambda \int_S f_{u_+} v. \tag{4.10}$$

Let us fix $x \in S$ and let us define a function $v : \overline{S} \rightarrow \mathbb{R}$ by the following formula

$$v(w) = \begin{cases} 1 & \text{for } x = w \\ 0 & \text{otherwise.} \end{cases}$$

Then we see from (4.10) that

$$-\Delta_{p(x),\omega} u(x) + q(x) |u(x)|^{p(x)-2} u(x) - \lambda f(x, u_+(x)) = 0.$$

Since $x \in S$ was fixed arbitrarily we get

$$-\Delta_{p(x),\omega} u(x) + q(x) |u(x)|^{p(x)-2} u(x) - \lambda f(x, u_+(x)) = 0 \text{ for all } x \in S.$$

Thus if $u \in A$ is a critical point the functional J it is a solution to problem (4.7). It is easy to see that every solution to problem (4.7) it is also a critical point to the functional J . ■

One important remark is in order as concerns the action functional J (given by the formula (4.8)). In the discrete boundary value problem one may take

either term connected with the difference operator or else with the nonlinearity as the leading one since all norms on a finite dimensional space are equivalent. In our case such approach is not possible because of the presence of the weight ω for which we cannot derive suitable inequalities as given in Section 3. and by the fact that we investigate the existence of positive solutions. Thus we shall use mainly term $\int_S \frac{1}{p} q |u|^p$ as the leading term in our investigations.

5. Existence of positive solutions

In this section we will seek positive solutions to problem (1.1). By a positive solution to problem (1.1) we mean such a function $u : \overline{S} \rightarrow \mathbb{R}$ which satisfies the given equation on \overline{S} , the boundary conditions on ∂S and it has only positive values on S . Positive solutions to (1.1) are investigated in the space A considered with the norm (3.4).

Put

$$u_+(x) = \max\{u(x), 0\}, \quad u_-(x) = \max\{-u(x), 0\} \quad \text{for all } x \in \overline{S}.$$

It is easy to see that for all $x \in \overline{S}$ we have

$$u_+(x), u_-(x) \geq 0;$$

$$u(x) = u_+(x) - u_-(x);$$

$$u_+(x) \cdot u_-(x) = 0;$$

$$|u(x)| = u_+(x) + u_-(x).$$

$$|u(x)_+| \leq |u(x)|.$$

Let us formulate an auxiliary result which plays an important role in proving all the existence results in this section. This result shows that any solution to (4.7) is in fact a positive solution and simultaneously it is the positive solution to (1.1). It may be viewed as a kind of a discrete maximum principle.

Assume that

(f.0) *The function f takes positive values for all $x \in S$ and all $t \geq 0$.*

Lemma 5.7 *Assume that (f.0) holds. Assume that $u \in A$ is a solution to problem (4.7). Then u has only positive values on S and moreover u is a positive solution to (1.1).*

Proof. A straightforward computation shows that for every $x, y \in \overline{S}$ the following inequality

$$(u(y) - u(x)) (u_-(y) - u_-(x)) \leq 0 \tag{5.11}$$

holds. Indeed,

$$\begin{aligned}
& (u(y) - u(x))(u_-(y) - u_-(x)) = \\
& ((u_+(y) - u_+(x)) - (u_-(y) - u_-(x)))(u_-(y) - u_-(x)) = \\
& - \left(u_+(y)u_-(x) + u_+(x)u_-(y) + (u_-(y) - u_-(x))^2 \right) \leq 0.
\end{aligned}$$

Assume that $u \in A$ is a solution to (4.7). Equating (4.9) to 0 and taking $v = u_-$ we obtain

$$\begin{aligned}
& \frac{1}{2} \sum_{x,y \in \overline{S}} |u(y) - u(x)|^{p(x)-2} (u(y) - u(x))(u_-(y) - u_-(x))\omega(x,y) = \\
& \lambda \sum_{x \in S} f(x, u_+(x))u_-(x) - \sum_{x \in S} q(x) |u(x)|^{p(x)-2} u(x)u_-(x).
\end{aligned} \tag{5.12}$$

Since f and q are functions with positive values only, $\lambda > 0$ and since

$$u(x)u_-(x) = (u_+(x) - u_-(x))u_-(x) = u_+(x)u_-(x) - (u_-(x))^2 \leq 0$$

the term on the right is non-negative. Due to (5.11) the term on the left is non-positive, therefore equation (5.12) holds if the both terms are equal zero, which leads to relation $u_-(x) = 0$ for all $x \in S$. Thus $u(x) = u_+(x)$ for all $x \in S$. Moreover $u(x) \neq 0$ for all $x \in S$. Indeed, assume that there exists $x_0 \in S$ such that $u(x_0) = 0$. Then by (1.1) we have

$$- \sum_{\{y \in \overline{S}: y \neq x_0\}} |u(y)|^{p(x_0)-1} \omega(x_0, y) = \lambda f(x_0, 0).$$

Since the term on the left is non-positive and the term on the right positive we have a contradiction. Thus $u(x) \neq 0$ for all $x \in S$, it follows that u is a positive solution to (1.1). ■

To show that problem (1.1) has positive solutions we need the following growth conditions

(f.1) *There exist functions $m_1, m_2 : S \rightarrow [2, +\infty)$ and functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : S \rightarrow (0, +\infty)$ such that*

$$\psi_1(x) + \varphi_1(x)t^{m_1(x)-1} \leq f(x, t) \leq \varphi_2(x)t^{m_2(x)-1} + \psi_2(x)$$

for all $x \in S$ and all $t \geq 0$.

Note that **(f.1)** implies **(f.0)**. Using the definition of F we get by integration

(F.1) *For functions $m_1, m_2 : S \rightarrow [2, +\infty)$ and functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : S \rightarrow (0, +\infty)$ satisfying **(f.1)** we have*

$$\psi_1(x)t + \frac{\varphi_1(x)}{m_1(x)}t^{m_1(x)} \leq F(x, t) \leq \frac{\varphi_2(x)}{m_2(x)}t^{m_2(x)} + \psi_2(x)t$$

for all $x \in S$ and all $t \geq 0$.

Let us introduce the following notations

$$\begin{aligned} q^- &= \min_{x \in S} q(x), & q^+ &= \max_{x \in S} q(x), \\ m_i^- &= \min_{x \in S} m(x), & m_i^+ &= \max_{x \in S} m(x), \quad i = 1, 2, \\ \varphi_1^- &= \min_{x \in S} \varphi_1(x), & \varphi_2^+ &= \max_{x \in S} \varphi_2(x), \\ \psi_1^- &= \min_{x \in S} \psi_1(x), & \psi_2^+ &= \max_{x \in S} \psi_2(x), \end{aligned}$$

where $m_1, m_2, \varphi_1, \varphi_2, \psi_1, \psi_2$ are functions defined in **(f.1)**.

Now we give an example to illustrate condition **(f.1)**.

Example 5.8 Let $m : S \rightarrow [2, +\infty)$ and $\varphi, \psi : S \rightarrow (0, +\infty)$. The function $f : S \times \mathbb{R} \rightarrow \mathbb{R}$ given by the formula

$$f(x, t) = (t + 1)^{1 - e^{-t^2} + m(x)} \left(\frac{2}{\pi} \arctan t + \varphi(x) \right) + |\sin t| + \psi(x) + 1$$

is a continuous function with only positive values for all $x \in S$ and all $t \geq 0$ and

$$1 + \psi(x) + t^{m(x)} \varphi(x) \leq f(x, t) \leq$$

$$2^{m(x)} t^{1+m(x)} (1 + \varphi(x)) + 2^{m(x)} (1 + \varphi(x)) + \psi(x) + 2.$$

for all $x \in S$ and all $t \geq 0$, so the growth conditions are satisfied with $m_1(x) = m(x)$, $m_2(x) = m(x) + 1$, $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = 2^{m(x)} (1 + \varphi(x))$, $\psi_1(x) = \psi(x) + 1$ and $\psi_2(x) = \psi(x) + 2$.

We will investigate the existence of positive solutions applying different methods, since depending on a relation between functions m_1, m_2 and p the functional J has different properties.

5.1 Results by the direct variational approach

We start with a case $m_2^+ < p^-$. Then for all $\lambda > 0$ the functional J is coercive and we can apply the direct variational method, Theorem 2.1. The case $m_2^+ = p^-$ is also undertaken, but in this case there is some restrictions on the parameter λ .

Theorem 5.9 Let $m_2^+ < p^-$. Assume that condition **(f.1)** is satisfied. Then for all $\lambda > 0$ problem (1.1) has at least one positive solution.

Proof. It suffices to show that the functional J is coercive on the set A so that to apply Theorem 2.1. By (a.7) we have

$$\sum_{x \in S} u_+(x) \leq \sum_{x \in S} |u_+(x)| \leq \sum_{x \in S} |u(x)| \leq |S| \max_{x \in S} |u(x)| \leq |S| |\overline{S}|^{\frac{1}{2}} \|u\|. \quad (5.13)$$

Therefore by (F.1) and (a.6) for sufficiently large $\|u\|$ we obtain

$$\sum_{x \in S} F(x, u_+(x)) \leq \frac{\varphi_2^+}{m_2} \left(|S| \|u\|^{m_2^+} + |S| \right) + \psi_2^+ |S| |\overline{S}|^{\frac{1}{2}} \|u\|^{m_2^+}. \quad (5.14)$$

By (a.4) and (5.14) for sufficiently large $\|u\|$ since $\int_S \frac{1}{p} \nabla_{p,\omega} u \circ \nabla_\omega u \geq 0$ we get

$$\begin{aligned} J(u) &\geq \int_S \frac{1}{p} q |u|^p - \lambda \int_S F_{u_+} \geq \\ &\frac{q^-}{p^+} \left(2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-} \|u\|^{p^-} - |S| \right) - \\ &\lambda \left(\frac{\varphi_2^+}{m_2} |S| \|u\|^{m_2^+} + \frac{\varphi_2^+}{m_2} |S| + \psi_2^+ |S| |\overline{S}|^{\frac{1}{2}} \|u\|^{m_2^+} \right). \end{aligned}$$

Since $m_2^+ < p^-$, so J is coercive on A .

The assumptions of Theorem 2.1 are satisfied and by Lemma 5.7 problem (1.1) has at least one positive solution. ■

Put

$$\lambda_1 := \frac{\frac{q^-}{p^+} 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-}}{\left(\frac{\varphi_2^+}{m_2} + \psi_2^+ |\overline{S}|^{\frac{1}{2}} \right) |S|}.$$

Remark 5.10 Let $m_2^+ = p^-$. Assume that condition (f.1) is satisfied. Then for all $\lambda \in (0, \lambda_1)$ problem (1.1) has at least one positive solution.

Proof. The assertion follows immediately by the proof of Theorem 5.9. ■

5..2 Result by the Ekeland variational principle

We have shown that the problem under consideration have at least one positive solution for all $\lambda > 0$ in case $m_2^+ < p^-$. In this subsection we apply Ekeland's variational principle in order to prove the existence of at least one positive solution for our problem for every parameter λ from some interval $(0, \lambda_2)$ with no inequality relation required on functions m_1 , m_2 and p apart from the assumption that $p^- \neq m_1^+$ at the expense of taking a suitable parameter interval.

Put

$$\lambda_2 := \frac{\frac{q^-}{p^+} 2^{-\frac{p^+}{2}} |\partial S|^{\frac{p^+}{2}} |\bar{S}|^{1-p^+} |\bar{S}|^{-\frac{p^+}{2}}}{\left(\frac{\varphi_2^+}{m_2^-} |\bar{S}|^{-\frac{m_2^-}{2}} + \psi_2^+ \right) |S|}$$

and

$$\Omega := \left\{ u \in A : \|u\| \leq |\bar{S}|^{-\frac{1}{2}} \right\}.$$

Theorem 5.11 *Let $p^- \neq m_1^+$. Assume that condition **(f.1)** is satisfied. Then for any $\lambda \in (0, \lambda_2)$ problem (1.1) has at least one positive solution.*

Proof. Let $\lambda \in (0, \lambda_2)$ be fixed. For all $u \in \Omega$ by **(a.7)** it follows that

$$|u(x)| \leq \max_{s \in S} |u(s)| \leq |\bar{S}|^{\frac{1}{2}} \|u\| \leq 1 \quad (5.15)$$

for all $x \in S$. By **(a.1)** for all $u \in \Omega$ we get

$$\sum_{x \in S} |u_+(x)|^{m_2(x)} \leq \sum_{x \in S} |u(x)|^{m_2(x)} \leq \sum_{x \in S} |u(x)|^{m_2^-} \leq |S| \|u\|^{m_2^-}. \quad (5.16)$$

By **(F.1)**, (5.16) and (5.13) for all $u \in \Omega$ we see that

$$\sum_{x \in S} F(x, u_+(x)) \leq \left(\frac{\varphi_2^+}{m_2^-} \|u\|^{m_2^-} + \psi_2^+ |\bar{S}|^{\frac{1}{2}} \|u\| \right) |S|. \quad (5.17)$$

Moreover by **(a.3)** for all $u \in \Omega$ we obtain

$$\sum_{x \in S} |u(x)|^{p(x)} \geq \sum_{x \in S} |u(x)|^{p^+} \geq 2^{-\frac{p^+}{2}} |\partial S|^{\frac{p^+}{2}} |\bar{S}|^{1-p^+} \|u\|^{p^+} \quad (5.18)$$

Therefore for all $u \in \partial\Omega$ by (5.18) and (5.17) we get

$$J(u) \geq \frac{q^-}{p^+} 2^{-\frac{p^+}{2}} |\partial S|^{\frac{p^+}{2}} |\bar{S}|^{1-p^+} |\bar{S}|^{-\frac{p^+}{2}} - \lambda \left(\frac{\varphi_2^+}{m_2^-} |\bar{S}|^{-\frac{m_2^-}{2}} + \psi_2^+ \right) |S|.$$

Thus for all $\lambda \in (0, \lambda_2)$ and for all $u \in \partial\Omega$ we have

$$J(u) > 0. \quad (5.19)$$

Since $\partial\Omega$ is a closed bounded set and since J is continuous, by the classical Weierstrass theorem we see that

$$\inf_{u \in \partial\Omega} J(u) = \min_{u \in \partial\Omega} J(u) > 0. \quad (5.20)$$

Put

$$t_0 := \min \left\{ 1, \left(\frac{2\lambda \left(\frac{\varphi_1^-}{m_1^+} + \psi_1^- \right) p^-}{\bar{\omega}^+ (2|S| + |\partial S| - 1) + 2q^+} \right)^{\frac{1}{p^- - m_1^+}} \right\}$$

and fix $t \in (0, t_0)$. Let $u_0 \in Int\Omega$ be such a function that $u_0(x_0) = t$ and $u_0(x) = 0$ for any $x \in S \setminus \{x_0\}$. First note that

$$\begin{aligned}
& \sum_{x, y \in S} |u_0(y) - u_0(x)|^{p(x)} = \\
& \sum_{x \in S} \left(|u_0(x_0) - u_0(x)|^{p(x)} + \sum_{y \neq x_0} |u_0(y) + u_0(x)|^{p(x)} \right) = \\
& \sum_{x \in S} (|t - u_0(x)|^{p(x)} + (|S| - 1) |u_0(x)|^{p(x)}) = \\
& (|S| - 1) t^{p(x_0)} + \sum_{x \neq x_0} t^{p(x)} \leq 2(|S| - 1) t^{p^-}.
\end{aligned} \tag{5.21}$$

Next we can observe that

$$\begin{aligned}
& \sum_{x \in \partial S} \sum_{y \in S} |u_0(y) - u_0(x)|^{p(x)} = \\
& \sum_{x \in \partial S} (|t - u_0(x)|^{p(x)} + (|S| - 1) |u_0(x)|^{p(x)}) \leq |\partial S| t^{p^-},
\end{aligned} \tag{5.22}$$

and

$$\sum_{x \in S} \sum_{y \in \partial S} |u_0(y) - u_0(x)|^{p(x)} = \sum_{x \in S} |u_0(x)|^{p(x)} = t^{p(x_0)} \leq t^{p^-}. \tag{5.23}$$

By (5.21), (5.22) and (5.23) we get

$$\begin{aligned}
& \sum_{x, y \in \bar{S}} |u_0(y) - u_0(x)|^{p(x)} \omega(x, y) \leq \bar{\omega}^+ \sum_{x, y \in S} |u_0(y) - u_0(x)|^{p(x)} + \\
& \bar{\omega}^+ \sum_{x \in \partial S} \sum_{y \in S} |u_0(y) - u_0(x)|^{p(x)} + \bar{\omega}^+ \sum_{x \in S} \sum_{y \in \partial S} |u_0(y) - u_0(x)|^{p(x)} \leq \\
& \bar{\omega}^+ (2|S| + |\partial S| - 1) t^{p^-}.
\end{aligned}$$

Therefore since $t \in (0, t_0)$ we have

$$J(u_0) \leq \frac{1}{2p^-} \bar{\omega}^+ (2|S| + |\partial S| - 1) t^{p^-} + \frac{q^+}{p^-} t^{p^-} - \lambda \left(\frac{\varphi_1^-}{m_1^+} + \psi_1^- \right) t^{m_1^+} < 0.$$

Hence

$$-\infty < \inf_{u \in Int\Omega} J(u) < 0. \tag{5.24}$$

By (5.20) and (5.24) we deduce that

$$\inf_{u \in Int\Omega} J(u) < \inf_{u \in \partial\Omega} J(u).$$

Remaining part of the proof is based on the relevant result from [10, 19] but since in the source mentioned it is derived for discrete BVP we decided to provide it in our setting for reader's convenience. Choose $\varepsilon > 0$ such that

$$\inf_{u \in \partial\Omega} J(u) - \inf_{u \in \text{Int}\Omega} J(u) > \varepsilon. \quad (5.25)$$

Applying Ekeland's variational principle, Theorem 5.11, to the functional $J : \Omega \rightarrow \mathbb{R}$ we find $u_\varepsilon \in \Omega$ such that

$$J(u_\varepsilon) \leq \inf_{u \in \Omega} J(u) + \varepsilon \quad (5.26)$$

and

$$J(u_\varepsilon) < J(u) + \varepsilon \|u - u_\varepsilon\| \quad \text{for all } u \in \Omega \text{ with } u \neq u_\varepsilon,$$

with $\varepsilon > 0$ satisfying (5.25). By (5.25) and (5.26) we get

$$J(u_\varepsilon) \leq \inf_{u \in \Omega} J(u) + \varepsilon \leq \inf_{u \in \text{Int}\Omega} J(u) + \varepsilon < \inf_{u \in \partial\Omega} J(u).$$

Thus $u_\varepsilon \in \text{Int}\Omega$. Note that u_ε is an argument of a minimum for the functional $\Phi : \Omega \rightarrow \mathbb{R}$ defined by

$$\Phi(u) := J(u) + \varepsilon \|u - u_\varepsilon\|,$$

so for any $v \in \Omega$ and a small enough real positive h we have

$$\frac{J(u_\varepsilon + hv) - J(u_\varepsilon)}{h} + \varepsilon \|v\| \geq 0.$$

Letting $h \rightarrow 0$ we obtain

$$\langle J'(u_\varepsilon), v \rangle + \varepsilon \|v\| \geq 0.$$

The above inequality holds for any $v \in \Omega$, so

$$|\langle J'(u_\varepsilon), v \rangle| \leq \varepsilon \|v\|.$$

Finally,

$$\|J'(u_\varepsilon)\| = \sup_{\|v\| \leq 1} \frac{|\langle J'(u_\varepsilon), v \rangle|}{\|v\|} \leq \varepsilon.$$

Putting $\varepsilon = \frac{1}{n}$ for sufficiently large natural n , we see that there exists a sequence $\{u_n\} \subset \text{Int}\Omega$ such that

$$J(u_n) \rightarrow \inf_{u \in \Omega} J(u) \quad \text{and} \quad J'(u_n) \rightarrow 0$$

as $n \rightarrow \infty$. The sequence $\{u_n\}$ is bounded in A , so there exists $v_0 \in A$ such that, up to a subsequence, $\{u_n\}$ converges to v_0 in A . Thus by the continuity of J and J' we have

$$J(v_0) = \inf_{u \in \Omega} J(u) \quad \text{and} \quad J'(v_0) = 0.$$

The above relations together with Theorem 4.6 and Lemma 5.7 imply that v_0 is a positive solution to (1.1). ■

6. Multiple solutions

6.1 Application of the Ekeland variational principle and mountain pass geometry

The relation $m_2^+ < p^-$ (studied by the direct variational approach) yields $m_1^- < p^-$. Using the technique described in [3] in case $m_1^- > \bar{p}^+$ we will show that problem (1.1) has at least two positive solutions for every parameter λ from interval $(0, \lambda_2)$. In this case the functional J is neither coercive nor anti-coercive (for the functional defined on a finite dimensional real Banach space the coercivity implies the Palais-Smale condition) but it satisfies the Palais-Smale condition. In [3] the Authors use the Ekeland Variational Principle together with the Mountain Pass Lemma.

Let us formulate an auxiliary result which provides the Palais-Smale condition.

Lemma 6.12 *Let $m_1^- > \bar{p}^+$. Assume that condition (f.1) is satisfied. Then the functional J satisfies the Palais-Smale condition.*

Proof. Assume that a sequence $\{u_n\}$ is such that $\{J(u_n)\}$ is bounded and $J'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since the space A is finite dimensional, it is enough to show that $\{u_n\}$ is bounded. Since $u_n(x) = u_n^+(x) - u_n^-(x)$ for all $n \in \mathbb{N}$ and all $x \in S$, it is enough to show that $\{u_n^+\}$ and $\{u_n^-\}$ are bounded.

Suppose that $\{u_n^-\}$ is unbounded. Then we may assume that there exists $N_0 > 0$ such that for all $n \geq N_0$ we have

$$\|u_n^-\| \geq q^- |S|. \quad (6.27)$$

Analogously as (5.11) we can show that

$$(u_+(y) - u_+(x))(u_-(y) - u_-(x)) \leq 0 \quad \text{for every } x, y \in \bar{S}. \quad (6.28)$$

Note also that

$$|u_-(y) - u_-(x)| \leq |u(y) - u(x)| \quad \text{for every } x, y \in \bar{S}. \quad (6.29)$$

Using (6.28) and (6.29) we obtain

$$\begin{aligned} & \sum_{x, y \in \bar{S}} |u(y) - u(x)|^{p(x)-2} (u(y) - u(x))(u_-(y) - u_-(x)) = \\ & \sum_{x, y \in \bar{S}} |u(y) - u(x)|^{p(x)-2} (u_+(y) - u_+(x))(u_-(y) - u_-(x)) - \\ & \sum_{x, y \in \bar{S}} |u(y) - u(x)|^{p(x)-2} (u_-(y) - u_-(x))(u_-(y) - u_-(x)) \leq \\ & - \sum_{x, y \in \bar{S}} |u(y) - u(x)|^{p(x)-2} (u_-(y) - u_-(x))^2 \leq - \sum_{x, y \in \bar{S}} |u_-(y) - u_-(x)|^{p(x)}. \end{aligned}$$

Moreover,

$$\begin{aligned}
& \sum_{x \in S} q(x) |u_n(x)|^{p(x)-2} u_n(x) u_n^-(x) = \\
& \sum_{x \in S} q(x) |u_n(x)|^{p(x)-2} (u_n^+(x) - u_n^-(x)) u_n^-(x) = \\
& - \sum_{x \in S} q(x) |u_n(x)|^{p(x)-2} (u_n^-(x))^2 = - \sum_{x \in S} q(x) |u_n^-(x)|^{p(x)}.
\end{aligned}$$

Bearing in mind (4.9) the above relations lead to

$$\begin{aligned}
\langle J'(u_n), u_n^- \rangle & \leq -\frac{1}{2} \sum_{x, y \in \bar{S}} |u_n^-(y) - u_n^-(x)|^{p(x)} \omega(x, y) - \\
& \sum_{x \in S} q(x) |u_n^-(x)|^{p(x)} - \lambda \sum_{x \in S} f(x, u_n^+(x)) u_n^-(x) \leq \\
& - \sum_{x \in S} q(x) |u_n^-(x)|^{p(x)}.
\end{aligned} \tag{6.30}$$

On the other hand by **(a.4)** we have

$$\sum_{x \in S} q(x) |u_n^-(x)|^{p(x)} \geq q^- \left(2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\bar{S}|^{1-p^-} \|u_n^-\|^{p^-} - |S| \right). \tag{6.31}$$

Thus by (6.30), (6.31) and the Schwartz inequality we deduce that

$$q^- \left(2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\bar{S}|^{1-p^-} \|u_n^-\|^{p^-} - |S| \right) \leq \langle J'(u_n), -u_n^- \rangle \leq \|J'(u_n)\| \cdot \|u_n^-\|.$$

In a consequence by (6.27) we get

$$\begin{aligned}
& q^- 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\bar{S}|^{1-p^-} \|u_n^-\|^{p^-} \leq \\
& \|J'(u_n)\| \cdot \|u_n^-\| + q^- |S| \leq \\
& \|J'(u_n)\| \cdot \|u_n^-\| + \|u_n^-\| \leq (\|J'(u_n)\| + 1) \|u_n^-\|.
\end{aligned}$$

By the above, since for some fixed $\varepsilon > 0$ there exists $N_1 \geq N_0$ such that $\|J'(u_n)\| < \varepsilon$ for every $n \geq N_1$, we get

$$\|u_n^-\|^{p^- - 1} \leq \frac{(\varepsilon + 1)}{q^- 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\bar{S}|^{1-p^-}}.$$

Contradiction. This means that $\{u_n^-\}$ is bounded.

It remains to show that $\{u_n^+\}$ is bounded. Suppose that $\{u_n^+\}$ is unbounded. By **(F.1)** and **(a.4)** for sufficiently large $\|u_n^+\|$ we obtain

$$\sum_{x \in S} F(x, u_n^+(x)) \geq \frac{\varphi_1^-}{m_1^+} \left(2^{-\frac{m_1^-}{2}} |\partial S|^{\frac{m_1^-}{2}} |\overline{S}|^{1-m_1^-} \|u_n^+\|^{m_1^-} - |S| \right) + \psi_1^- \sum_{x \in S} u_n^+(x). \quad (6.32)$$

By **(a.5)**, **(a.6)** and (6.32) we get

$$\begin{aligned} J(u_n) &\leq \frac{1}{2\overline{p}^-} \overline{\omega}^+ \left(2^{\overline{p}^+} |\overline{S}| |S| \|u_n^+ - u_n^-\|^{\overline{p}^+} + |\overline{S}|^2 \right) + \frac{q^+}{p^-} \left(|S| \|u_n^+ - u_n^-\|^{p^+} + |S| \right) - \\ &\lambda \left(\frac{\varphi_1^-}{m_1^+} 2^{-\frac{m_1^-}{2}} |\partial S|^{\frac{m_1^-}{2}} |\overline{S}|^{1-m_1^-} \|u_n^+\|^{m_1^-} - \frac{\varphi_1^-}{m_1^+} |S| + \psi_1^- \sum_{x \in S} u_n^+(x) \right) \leq \\ &\frac{1}{2\overline{p}^-} \overline{\omega}^+ \left(2^{\overline{p}^+} (\overline{p}^+ - 1) |\overline{S}| |S| \left(\|u_n^+\|^{\overline{p}^+} + \|u_n^-\|^{\overline{p}^+} \right) + |\overline{S}|^2 \right) + \\ &\frac{q^+}{p^-} \left(2^{p^+ - 1} |S| \left(\|u_n^+\|^{\overline{p}^+} + \|u_n^-\|^{\overline{p}^+} \right) + |S| \right) - \\ &\lambda \left(\frac{\varphi_1^-}{m_1^+} 2^{-\frac{m_1^-}{2}} |\partial S|^{\frac{m_1^-}{2}} |\overline{S}|^{1-m_1^-} \|u_n^+\|^{m_1^-} - \frac{\varphi_1^-}{m_1^+} |S| + \psi_1^- \sum_{x \in S} u_n^+(x) \right). \end{aligned}$$

Since $m_1^- > \overline{p}^+$ and $\{u_n^+\}$ is unbounded and $\{u_n^-\}$ is bounded, so $J(u_n) \rightarrow -\infty$ as $\|u_n^+\| \rightarrow \infty$. Thus we obtain a contradiction with the assumption that $\{J(u_n)\}$ is bounded, so it follows that $\{u_n^+\}$ is bounded. Hence $\{u_n\}$ is bounded. \blacksquare

At the end of Section 2. we indicated that results from [3] could be applied in order to get multiple solutions. Now we are going to apply these for the problem under consideration. Recall that

$$\lambda_2 := \frac{\frac{q^-}{p^+} 2^{-\frac{p^+}{2}} |\partial S|^{\frac{p^+}{2}} |\overline{S}|^{1-p^+} |\overline{S}|^{-\frac{p^+}{2}}}{\left(\frac{\varphi_2^+}{m_2^-} |\overline{S}|^{-\frac{m_2^-}{2}} + \psi_2^+ \right) |S|}$$

and

$$\Omega := \left\{ u \in A : \|u\| \leq |\overline{S}|^{-\frac{1}{2}} \right\}.$$

Theorem 6.13 *Let $m_1^- > \overline{p}^+$. Assume that condition **(f.1)** is satisfied. Then for any $\lambda \in (0, \lambda_2)$ problem (1.1) has at least two distinct positive solutions.*

Proof. By Lemma 6.12 the functional J satisfies the Palais-Smale condition. Let $\lambda \in (0, \lambda_2)$ be fixed. Note that $\overline{\text{Int}\Omega} = \Omega$. Since $0 \in \text{Int}\Omega$, so by (5.20) we deduce that

$$\min_{u \in \Omega} J(u) \leq J(0) = 0 < \min_{u \in \partial\Omega} J(u). \quad (6.33)$$

Thus we have relation (2.3) satisfied.

Let $u_\xi \in A$ be defined as follows: $u_\xi(x) = \xi$ for all $x \in S$ and $u_\xi(0) = 0$ for all $x \in \partial S$. Then for $\xi > 1$ we have

$$\begin{aligned} \sum_{x,y \in \overline{S}} |u(y) - u(x)|^{p(x)} &= \\ \sum_{x \in \partial S} \sum_{y \in S} |u(y) - u(x)|^{p(x)} + \sum_{x \in S} \sum_{y \in \partial S} |u(y) - u(x)|^{p(x)} &= \\ \sum_{x \in \partial S} |\xi - u(x)|^{p(x)} + \sum_{x \in S} |u(x)|^{p(x)} &\leq (|\partial S| + |S|) \xi^{\overline{p}^+}. \end{aligned}$$

Therefore

$$J(u_\xi) \leq \frac{1}{2\overline{p}^-} (|\partial S| + |S|) |\xi|^{\overline{p}^+} \overline{\omega}^+ + \frac{q^+}{p^-} |S| |\xi|^{p^+} - \lambda |S| \left(\frac{\varphi_1^- \xi^{m_1^-}}{m_1^+} + \psi_1^- \xi \right).$$

Since $m_1^- > \overline{p}^+$, then $\lim_{\xi \rightarrow \infty} J(u_\xi) = -\infty$, so there exists ξ_0 such that $u_{\xi_0} \in A \setminus \Omega$ and

$$J(u_{\xi_0}) < \min_{u \in \partial \Omega} J(u).$$

Thus by the remarks contained at the end of Section 2. provide the assertion. \blacksquare

Now we proceed with some suggestion of the alternative proof of Theorem 6.13 by using Corollary 3.2. from [4] which says that if a functional satisfying the Palais-Smale condition is unbounded from below and has a local minimum then it has another critical point. From the proof of Theorem 6.13 it follows that the functional J has a local minimum on a ball

$$\Omega := \left\{ u \in A : \|u\| \leq |\overline{S}|^{-\frac{1}{2}} \right\}.$$

From the proof of Lemma 6.12 we see that the functional J satisfies the Palais-Smale condition. Moreover, from the proof of Theorem 6.13 it follows that the functional J is unbounded from below. To conclude, both methods require similar calculations to be performed since both abstract results are based on similar tools.

6..2 Application of the mountain pass geometry and Karush-Kuhn-Tucker theorem

When relation (2.3) is not satisfied we cannot use the arguments mentioned in Theorem 6.13 since this condition is crucial since one solution is obtained via the Ekeland's Principle and it must lie in the interior of the set, while the second one it reached through the Mountain Pass Geometry. But we have some other tools

at hand. So in this subsection we apply Karush-Kuhn-Tucker theorem together with the mountain pass geometry in order to obtain the existence of at least two distinct positive solutions with at least one solution outside the unit ball. The first minimizer we find using the Karush-Kuhn-Tucker conditions. The second minimizer there exists by the mountain pass technique. Thus our ideas are related to those contained in [3] since one solution is reached by the mountain pass technique and the second by some other technique which provides that it lies in the interior of the ball.

Put

$$\gamma_0 := 2^{\frac{1}{2}} |\partial S|^{\frac{1}{2}} |\overline{S}|.$$

Theorem 6.14 *Let $m_1^- > \overline{p}^+$. Assume that condition (f.1) is satisfied. Let us choose $\gamma > \gamma_0$ and put*

$$\lambda_3 := \frac{q^- 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-} \gamma^{p^-} - q^- |S|}{\left(\varphi_2^+ \gamma^{m_2^+} + \varphi_2^+ + \psi_2^+ |\overline{S}|^{\frac{1}{2}} \gamma \right) |S|}.$$

Then for any $\lambda \in (0, \lambda_3)$ problem (1.1) has at least two distinct positive solutions with at least one positive solution outside the unit ball.

Proof. Let $\lambda \in (0, \lambda_3)$ be fixed. Note that $\gamma_0 > 1$. Put

$$\Omega_1 := \{u \in A : \|u\| \leq \gamma\}; \quad \Omega_2 := \{u \in A : \|u\| \geq \zeta\},$$

where $\zeta \in (1, \gamma)$. Put

$$\Omega := \Omega_1 \cap \Omega_2.$$

The set Ω is bounded and closed, so the classical Weierstrass theorem implies that the functional J attains a minimum in Ω . Assume that $u_0 \in A$ is a local minimizer of J in Ω . We will show, by a contradiction, that u_0 is the element required by the Mountain Pass Lemma, that is $u_0 \notin \partial\Omega_1$. Suppose otherwise, that $u_0 \in \partial\Omega_1$.

Applying the Karush-Kuhn-Tucker theorem, Theorem 2.4, to the problem

$$\min_{u \in A} J(u)$$

subject to the constraints

$$\begin{cases} \|u\|^2 - \gamma^2 \leq 0; \\ \zeta^2 - \|u\|^2 \leq 0, \end{cases}$$

we deduce that there exist constants $\kappa, \sigma, \vartheta \geq 0$ do not vanish simultaneously such that

$$\sigma(\|u_0\|^2 - \gamma^2) = 0 \quad \text{and} \quad \vartheta(\zeta^2 - \|u_0\|^2) = 0 \quad (6.34)$$

and

$$\kappa \langle J'(u_0), v \rangle + \sigma \langle u_0, v \rangle - \vartheta \langle u_0, v \rangle = 0 \quad (6.35)$$

for all $v \in A$.

The set $\left\{u \in \Omega : \|u\|^2 - \gamma^2 \leq 0 \text{ and } \zeta^2 - \|u\|^2 \leq 0\right\}$ has a non-empty interior, so we may put $\kappa = 1$. By (6.34) we deduce that $\vartheta = 0$, since $\|u_0\| = \gamma \neq \zeta$ and so $\zeta^2 - \|u_0\|^2 \neq 0$. Now suppose that $\sigma > 0$. Then by (6.35) and (4.9) we get

$$\begin{aligned} & \frac{1}{2} \sum_{x,y \in \overline{S}} |u_0(y) - u_0(x)|^{p(x)-2} (u_0(y) - u_0(x))(v(y) - v(x))\omega(x,y) + \\ & \sum_{x \in S} q(x) |u_0(x)|^{p(x)-2} u_0(x)v(x) - \lambda \sum_{x \in S} f(x, u_0^+(x))v(x) + \sigma \sum_{x \in S} \langle u_0(x), v(x) \rangle = 0. \end{aligned}$$

for all $v \in A$. Taking $v = u_0$ we see that

$$\begin{aligned} & \frac{1}{2} \sum_{x,y \in \overline{S}} |u_0(y) - u_0(x)|^{p(x)} \omega(x,y) + \sum_{x \in S} q(x) |u_0(x)|^{p(x)} + \sigma \|u_0\|^2 = \\ & \lambda \sum_{x \in S} f(x, u_0^+(x))u_0(x). \end{aligned} \tag{6.36}$$

Since $u_0 \in \partial\Omega_1$, we see that $\|u_0\| = \gamma$. Thus by **(a.4)** we obtain

$$\begin{aligned} & \frac{1}{2} \sum_{x,y \in \overline{S}} |u_0(y) - u_0(x)|^{p(x)} \omega(x,y) + \sum_{x \in S} q(x) |u_0(x)|^{p(x)} + \sigma \|u_0\|^2 \geq \\ & q^- \left(2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-} \gamma^{p^-} - |S| \right) + \sigma \gamma^2. \end{aligned} \tag{6.37}$$

Note that the term on the right is positive, since $\gamma > \gamma_0$.

By **(a.6)** we get

$$\sum_{x \in S} |u_0^+(x)|^{m_2(x)} \leq \sum_{x \in S} |u_0(x)|^{m_2(x)} \leq |S| \|u_0\|^{m_2^+} + |S| = \left(\gamma^{m_2^+} + 1 \right) |S|. \tag{6.38}$$

Moreover by (5.13) we obtain

$$\sum_{x \in S} u_0^+(x) \leq |S| |\overline{S}|^{\frac{1}{2}} \|u_0\| = |S| |\overline{S}|^{\frac{1}{2}} \gamma. \tag{6.39}$$

By **(f.1)**, **(6.38)** and (6.39) we infer that

$$\sum_{x \in S} f(x, u_0^+(x))u_0(x) \leq \sum_{x \in S} f(x, u_0^+(x))u_0^+(x) \leq \left(\varphi_2^+ \gamma^{m_2^+} + \varphi_2^+ + \psi_2^+ |\overline{S}|^{\frac{1}{2}} \gamma \right) |S|. \tag{6.40}$$

Thus by (6.36), (6.37) and (6.40) we get

$$q^- 2^{-\frac{p^-}{2}} |\partial S|^{\frac{p^-}{2}} |\overline{S}|^{1-p^-} \gamma^{p^-} - q^- |S| + \sigma \gamma \leq \lambda \left(\varphi_2^+ \gamma^{m_2^+} + \varphi_2^+ + \psi_2^+ |\overline{S}|^{\frac{1}{2}} \gamma \right) |S|.$$

A contradiction with the assumption $\lambda \in (0, \lambda_3)$.

Eventually $\vartheta = \sigma = 0$ and $\kappa \neq 0$. Therefore $u_0 \notin \partial\Omega_1$, so

$$J(u_0) < \min_{u \in \partial\Omega_1} J(u).$$

Moreover (see the proof of Theorem 6.13) there exists $u_{\xi_0} \in A \setminus \Omega_1$ such that $J(u_{\xi_0}) < \min_{u \in \partial\Omega_1} J(u)$.

By Lemma 6.12 and Lemma 2.2 we obtain a critical value of the functional J for some $u^* \in A$. Moreover u_0 and u^* are two different critical points of J and therefore by Lemma 5.7 there are two distinct positive solutions to (1.1). ■

Remark 6.15 *We note that the closer γ to γ_0 in the above theorem, the eigenvalue interval becomes larger.*

Example 6.16 *Let $S = \{x_1, x_2, x_3\}$, $\partial S = \{x_4, x_5, x_6\}$ and put $\omega(x_1, x_2) = \omega(x_2, x_3) = \omega(x_3, x_1) = \omega(x_1, x_4) = \omega(x_2, x_5) = \omega(x_3, x_6) = a > 0$. Let $p : \overline{S} \rightarrow [2, +\infty)$, $m : S \rightarrow [2, +\infty)$ and $q, \varphi, \psi : S \rightarrow (0, +\infty)$ are given by the formulas*

$$m(x_i) = 2i^2; \quad \varphi(x_i) = 3i - 1; \quad \psi(x_i) = i; \quad q(x_i) = e^{i+31} \text{ for } i = 1, 2, 3$$

and

$$p(x_i) = i + 3 \text{ for } i = 1, 2, \dots, 6.$$

We have shown in the example 5.8 that the function $f : S \times \mathbb{R} \rightarrow \mathbb{R}$ given by the formula

$$f(x, t) = (t + 1)^{1 - e^{-t^2} + m(x)} \left(\frac{2}{\pi} \arctan t + \varphi(x) \right) + |\sin t| + \psi(x) + 1$$

satisfies condition (f.1) with $m_1(x) = m(x)$, $m_2(x) = m(x) + 1$, $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = 2^{m(x)}(1 + \varphi(x))$, $\psi_1(x) = \psi(x) + 1$ and $\psi_2(x) = \psi(x) + 2$.

Note that $18 = m_2^+ \not\leq p^- = 4$, so assumption of Theorem 5.9 are not satisfied, but $18 = m_1^+ \neq p^- = 4$ so by Theorem 5.11 we have that for all $\lambda \in (0, \lambda_2)$, where $\lambda_2 = 6.3823 \times 10^{-4}$, problem (1.1) has at least one positive solution.

Now put $m(x_i) = 10i$ for $i = 1, 2, 3$. Then $10 = m_1^- > \overline{p}^+ = 9$ (assumption of Theorem 5.9 are not satisfied) and by Theorem 6.13 problem (1.1) has at least two distinct positive solutions for all $\lambda \in (0, \lambda_2)$, where $\lambda_2 = 0.816$. Put $\gamma = 14.7$. Then by Theorem 6.14 ($\gamma > \gamma_0 = 14.697$) we have that for all $\lambda \in (0, \lambda_3)$, where $\lambda_3 = 2.7065 \times 10^{-20}$, problem (1.1) has at least two distinct positive solutions with at least one positive solution outside the unit ball.

If we put $m(x_i) = 2 \sin^2 \frac{\pi i}{3}$, we have by Theorem 5.9 the existence of at least one positive solution for problem (1.1) for all $\lambda > 0$, since $3 = m_2^+ < p^- = 4$.

References

- [1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.

- [2] C. Bereanu, P. Jebelean, C. Serban, *Periodic and Neumann problems for discrete $p(\cdot)$ -Laplacian*, J. Math. Anal. Appl. 399 (2013), 75-87.
- [3] C. Bereanu, P. Jebelean and J. Mawhin, *Multiple solutions for Neumann and periodic problems with singular ϕ -Laplacian*, J. Funct. Anal. 261 (2011), 3226-3246.
- [4] G. Bonanno, P. Candito, G. D'Agui, *Variational methods on finite dimensional Banach spaces and discrete problems*. Adv. Nonlinear Stud. 14 (2014), no. 4, 915-939.
- [5] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer New York Dordrecht Heidelberg London, 2011.
- [6] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. 66 (2006), 1383-1406.
- [7] S.-Y. Chung and J.-H. Park, *Multiple solutions to discrete boundary value problems for the p -Laplacian with potential terms on finite graphs*, Bull. Korean Math. Soc. 52 (2015), No. 5, 1517-1533.
- [8] D.G. De Figueiredo, *Lectures on The Ekeland Variational Principle with Applications and Detours*. Bombay: Tata Institute of Fundamental Research, 1989.
- [9] X.L. Fan, D. Zhang, *Existence of solutions for $p(x)$ -Laplacian Dirichlet problem*, Nonl. Analysis 52 (2003), 1843-1852.
- [10] M. Galewski, R. Wieteska, *Existence and multiplicity of positive solutions for discrete anisotropic equations*, Turk. J. Math., 38 (2014), 297-310.
- [11] I.V. Girsanow, *Lectures on Mathematical Theory of Extremum Problems*, Springer Verlag Berlin Hedelberg New York, 1972.
- [12] P. Harjulehto, P. Hästö, U. V. Le and M. Nuortio, *Overview of differential equations with non-standard growth*, Nonlinear Anal. 72 (2010), 4551-4574.
- [13] Y. Jabri, *The Mountain Pass Theorem, Variants, Generalizations and some Applications*, Cambridge University Press, 2003.
- [14] A. Iannizzotto, V. Radulescu, *Positive homoclinic solutions for the discrete p -Laplacian with a coercive potential*, Differential and Integral Equations 27 (2014), 35-44.
- [15] J.-H. Kim, *The (p, ω) -Laplacian operators on nonlinear networks*, Ph. D. Thesis, University of Sogang at Korea.
- [16] A. Kristály, V. Radulescu, C. Varga, *Variational Principles in Mathematical Physics, Geometry and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics (No. 136), Cambridge University Press, Cambridge, 2010.

- [17] N. Marcu, G. Molica Bisci, *Existence and multiplicity of solutions for non-linear discrete inclusions*, Electron. J. Differential Equations 2012, No. 192, 13 pp.
- [18] Mawhin J. Problèmes de Dirichlet variationnels non linéaires. Les Presses de l'Université de Montréal, 1987.
- [19] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems. J. Difference Equ. Appl. 15 (2009), no. 6, 557–567.
- [20] J.H. Park, S.Y. Chung, Positive solutions for discrete boundary value problems involving the p-Laplacian with potential terms, Comput. Math. Appl. 61 (2011), 17-29.
- [21] J.H. Park, On a resonance problem with the discrete p-Laplacian on finite graphs, Nonlinear Anal. 74 (2011), 6662-6675.
- [22] C. Șerban, *Existence of solutions for discrete p-Laplacian with potential boundary conditions*, J. Difference Equ. Appl. 19 (2013), 527-537.
- [23] C. Șerban, *Multiplicity of solutions for periodic and Neumann problems involving the discrete $p(\cdot)$ -Laplacian*, Taiwanese J. Math. 17 (4) (2013), 1425-1439
- [24] M. Struwe, *Variational Methods*, Springer, Berlin, 1996.
- [25] M. Růžička, *Electrorheological fluids: Modelling and Mathematical Theory*, in: Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.
- [26] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv. 29 (1987), 33-66.

Marek Galewski, Renata Wieteska
 Institute of Mathematics,
 Technical University of Lodz,
 Wolczanska 215, 90-924 Lodz, Poland,
 marek.galewski@p.lodz.pl, renata.wieteska@p.lodz.pl